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View Factor for Two Spheres

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Abstract — *The concept of a view factor has recently been proposed to describe the fraction of leakage neutrons from a spherical nuclear assembly that will enter another coupled assembly. Earlier applications used an approximate form of this quantity for the transport of heat and radiation between two spheres. The same approximate form was employed in the case of coupled nuclear assemblies. We show that it is possible to obtain an exact expression for this quantity that eliminates the need for such an approximation.*

Keywords — *View factor, coupled spherical assemblies, Felske formula.*

Note — *Some figures may be in color only in the electronic version.*

I. INTRODUCTION

The concept of a view factor has applications in the transport of heat and radiation. The view factor is defined by a four-dimensional integral as described in a later section. For the case of two spheres, an approximate form of this quantity (referred to as Felske's approximation) has been in use in heat transport studies.[1] In a recent paper dealing with the study of initiation probability in coupled spherical assemblies, Prinja et al.[2] employ the Felske approximation to obtain the view factor. An earlier paper by O'Rourke and Prinja[3] discusses various methods such as Felske's approximate expression and a detailed Monte Carlo approach for obtaining this quantity. The integrals involved in the calculation of the view factor are considered to be too complicated[3] for exact analytical evaluation, and hence, the approximate form or Monte Carlo calculations have to be employed for this purpose.

This letter shows that by choosing the variables suitably and by rewriting the integral, it is possible to reduce the four-dimensional integral to a simple one-dimensional integral. For spheres with equal radii, an analytical expression in terms of complete elliptic integrals is obtained (which reduces to an elementary integral for two equal spheres in contact with one another). For the general case of unequal spheres, the one-dimensional integral is easily evaluated using low-order Gauss quadrature. Thus, it becomes possible to obtain accurate values of the view factor for spherical systems without the need for approximate forms or more involved Monte Carlo calculations. Results obtained by the present method and comparisons with the Felske approximation are presented.

II. DEFINITION OF THE VIEW FACTOR

The view factor is defined[3] by the following integral over the surfaces of the two bodies under consideration:

$$F_{1\to 2} = \frac{1}{A_1}\int\int dA_1 dA_2 \frac{\cos\beta_1 \cos\beta_2}{\pi L^2} \quad , \tag{1}$$

where β_1 and β_2 = angles that the line joining the points on the two surface makes with their respective normals and L is the length of this line (Fig. 1); $A_1 = 4\pi R_1^2$ = surface area of sphere 1; $A_2 = 4\pi R_2^2$ = surface area of sphere 2.

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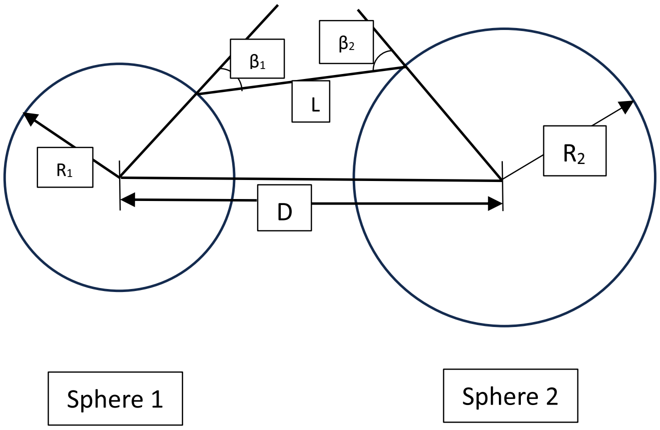

Fig. 1. Illustration of the quantities appearing in the definition of the view factor in Eq. (1). It may be noted that the flight path and point of entry in sphere 2 are not necessarily coplanar with the centers of the spheres and the point of exit since Eq. (1) implies integration over all points of the surfaces of each of the two spheres.

By writing

$$\frac{\cos\beta_2 dA_2}{L^2} = d\Omega \tag{1a}$$

for the solid angle element subtended by the area dA_2 at a point at dA_1 and by interpreting

$$\cos\beta_1 dA_1 = dS \tag{1b}$$

as an element of area perpendicular to the direction of the line joining the points near dA_1 and dA_2, which is the same as the direction of flight of a neutron (i.e., as the projection of the element dA_1 in a plane perpendicular to Ω), we can rewrite the integral in Eq. (1) in the form

$$F_{1\to 2} = \frac{1}{\pi A_1}\int d\Omega \int dS \ , \tag{2}$$

where the integration over Ω is over all directions (effectively those directions in which any ray intersects both spheres). Obviously, the transformation implied by Eqs. (1a) and (1b) leads to Eq. (2), but how do we physically interpret this transformation to proceed further? For this, we refer to Fig. 2a. The figure shows two spheres having radii R_1 and R_2 ($R_1 \le R_2$) with centers located at (0, 0, 0) and (0, 0, D) of the coordinate system (X, Y, Z). Since we are integrating first with respect to S, where S represents an area perpendicular to Ω, we fix a direction Ω [i.e., (θ, φ)] and project the spheres in a plane P perpendicular to Ω. The projections in the plane are circles.

Plane P is shown in detail in Fig. 2b. A coordinate system (x, y) [different from the original system (X, Y, Z)] is chosen in this plane such that the circles corresponding to spheres 1 and 2 are located at the origin and at a distance d away on the x-axis, respectively. We consider all possible rays in the direction Ω starting uniformly from the inside of circle 1, which is the projection of sphere 1. That is, we consider the cylindrical pencil of paths starting from circle 1 in direction Ω. Since we are interested in only those paths that begin from sphere 1 and enter sphere 2 for the purpose of integration over S, it is clear that only those rays starting from the area that is common to the two circles in plane P contribute to the integral over S. In other words, the integral over S is simply the common area enclosed by these circles; i.e., it is the sum of the areas of regions S_1 and S_2, (as marked in Fig. 2b) denoted by S_1 and S_2, respectively.

To integrate over the direction variable Ω, we note that the area $S_1 + S_2$ depends only on the azimuthal angle θ, and hence, this integration reduces to an integration over θ alone, and integration over φ simply contributes the factor 2π. Thus, we can write $d\Omega = 2\pi \sin\theta$. Since the integration over S is restricted to all rays in the direction Ω that pass through the first sphere and intersect the second sphere, it sets the upper limit of integration over the angle θ. Figure 3 shows the upper limit on the integration with respect to θ, namely, ψ_2. Above this value of θ, there is no intersection with sphere 2, and the integrand is zero. Further, referring to Fig. 4, we see that for $\theta < \psi_1$, the projection of sphere 1 lies completely within the projection of sphere 2, and hence, for $\theta < \psi_1$, the integrand is a constant. Thus, the integration over θ splits into two parts, namely, from 0 to ψ_1 and from ψ_1 to ψ_2 as discussed in greater detail in Sec. III.

III. CALCULATION OF THE VIEW FACTOR

For the case of two spheres, having radii R_1 and R_2 ($R_1 \le R_2$) with centers located at (0, 0, 0) and (0, 0, D), we can calculate the view factor by projecting the spheres in a plane perpendicular to the direction Ω of the rays. Each of the projections is a circle as shown in Fig. 2b. The integral over S, for the direction Ω, is given by the overlapping area of these circles. This area can be written as the sum of the areas of the shaded region (between the common chord and circle 1 and marked as S_1) and the corresponding white (unshaded) region (between the common chord and circle 2 and marked as S_2). When integrated over the direction variable Ω, we obtain the required quantity $F_{1\to 2}$. The areas S_1 and S_2 may be calculated as follows.

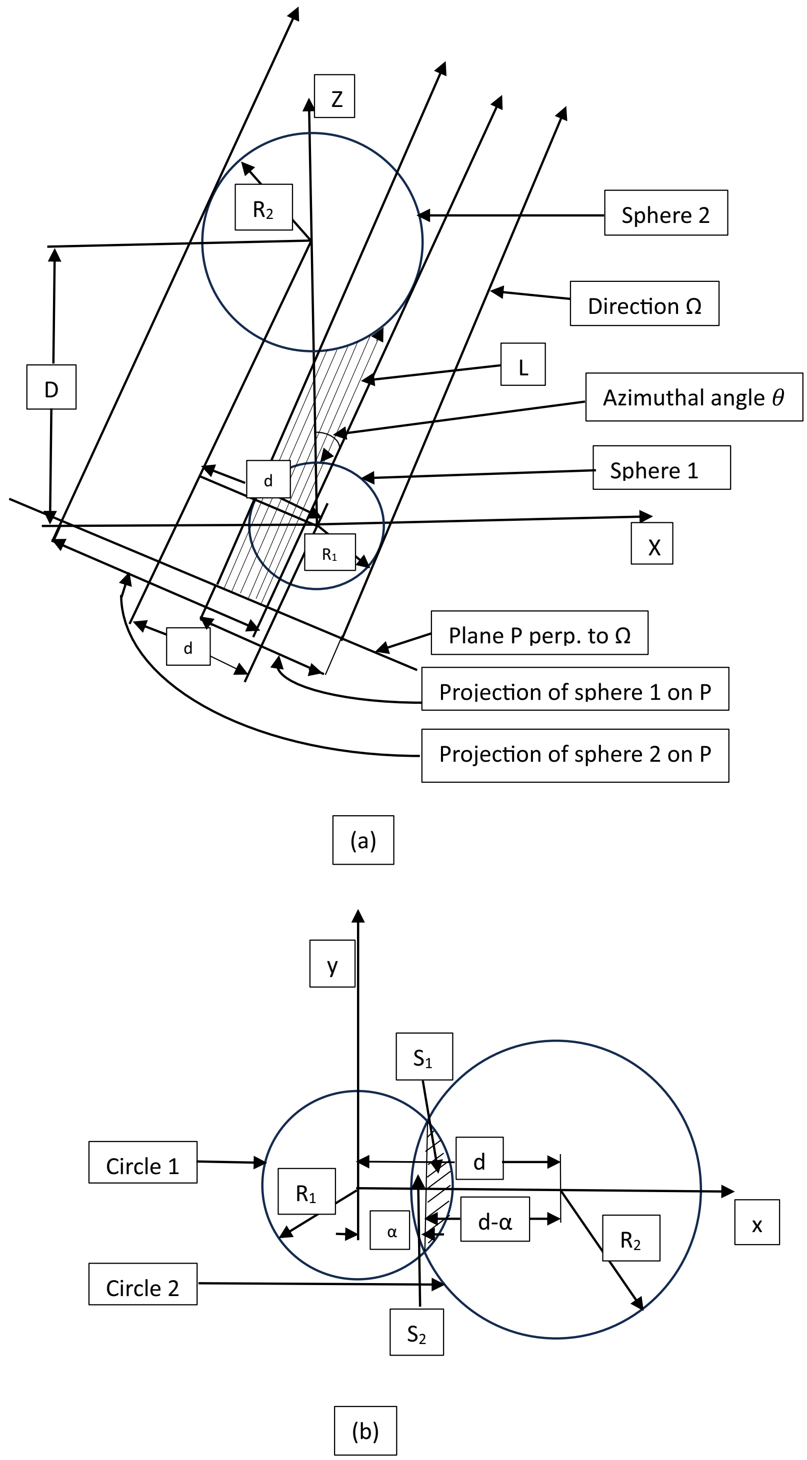

Fig. 2. (a) Illustration of the method of obtaining the transformed integral for the view factor [Eq. (2)] in the (X, Z) plane. Sphere 1 has radius R_1, and its center is at the origin while sphere 2 has radius R_2, and its center is at $(0, 0, D)$, i.e., at a distance D away from the origin on the Z-axis. Direction Ω (θ, φ) is chosen, and the spheres are projected in plane P perpendicular to Ω. The projections are circles, and d is the distance between their centers. In the case shown, the projections of the spheres have an overlapping area as some of the rays pass through both spheres. Integration over S [i.e., dS in Eq. (2)] is simply the area of this overlapping region. (b) A view is shown of the projection of the spheres in P (P is shown in the plane of the paper in this figure) for the purpose of calculating the overlapping area. A coordinate system (x, y) [different from the original system (X, Y, Z)] is chosen in this plane such that the circles corresponding to spheres 1 and 2 are located at the origin and at a distance d away on the x-axis, respectively. α and d-α are the distances of the line joining the intersection points of the circles from the centers of spheres 1 and 2, respectively.

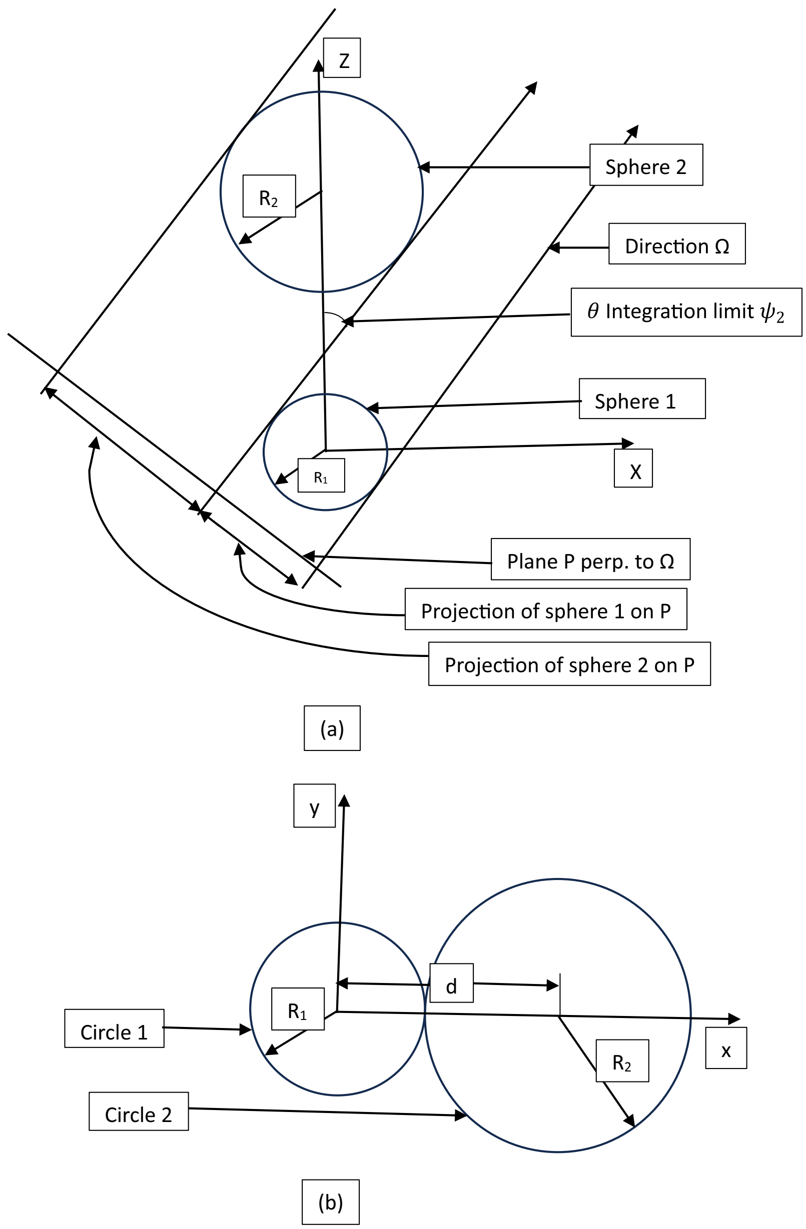

Fig. 3. Illustration of the angle ψ_2, the upper limit on integration over θ. Beyond this, there is no intersection with sphere 2, and there is no overlap of the projections of the spheres: (a) is the representation in the *X-Z* plane and (b) is in the plane of projection P.

Since in the system of coordinates chosen in plane P, the centers of the circles are located at $(0,0)$ and $(d,0)$, we may write equations for the circles as

$$x^2 + y^2 = R_1^2 \tag{3a}$$

$$(x-d)^2 + y^2 = R_2^2 \ . \tag{3b}$$

To find the areas S_1 (and S_2), we need to know the distance between the center of circle 1 (circle 2) and the common chord, denoted by α (γ). The distance α is simply the x coordinate of the points of intersection and is obtained by solving Eqs. (3a) and (3b). The equations are easily solved by eliminating y^2, and we get

$$x = \frac{d^2 + R_1^2 - R_2^2}{2d} = \alpha \ . \tag{4}$$

Since d is related to the angle θ that the direction of the rays makes with the line joining the centers of the two spheres as follows:

$$d = D \sin\theta \ , \tag{5}$$

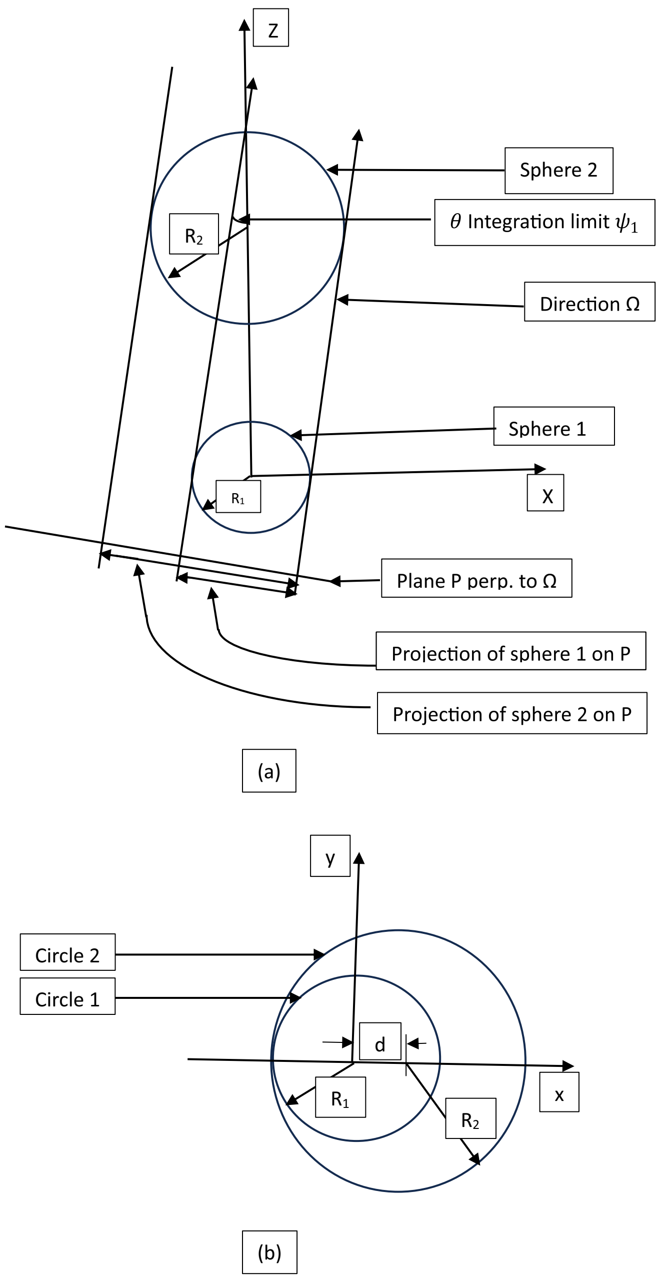

Fig. 4. Illustration of the limit ψ_1 for the integration over θ. The integrand is constant between $\theta = 0$ to $\theta = \psi_1$ and is simply the area of the projection of sphere 1: (a) is the view in the *XZ* plane and (b) is the view in the plane of projection P.

we can write

$$\alpha = \frac{D^2 \sin^2\theta + R_1^2 - R_2^2}{2D \sin\theta} \ , \tag{6a}$$

and since the distance $\gamma = d - \alpha$, we get

$$\gamma = d - \alpha = \frac{D^2 \sin^2\theta + R_2^2 - R_1^2}{2D \sin\theta} \ . \tag{6b}$$

Thus, the two parts of the overlapping regions S_1 and S_2 have areas given by

$$S_1 = 2\int_{\alpha}^{R_1} \sqrt{R_1^2 - x^2}dx$$
$$= R_1^2\left[(\pi/2 - \arcsin(\alpha/R_1)) - \alpha/R_1\sqrt{1-(\alpha/R_1)^2}\right] \tag{7a}$$

and

$$S_2 = 2\int_{d-R_2}^{\alpha} \sqrt{R_2^2 - (x-d)^2}dx$$
$$= R_2^2\left[(\pi/2 - \arcsin(\gamma/R_2)) - \gamma/R_2\sqrt{1-(\gamma/R_2)^2}\right] \ . \tag{7b}$$

As may be seen from Eqs. (6) and (7), S_1 and S_2 depend on only θ and not on the azimuthal angle φ. We can write $d\Omega = 2\pi \sin\theta$, and hence, the expression for the view factor becomes

$$F_{1\to 2} = \frac{1}{2\pi R_1^2}\int_0^{\psi_2}(S_1 + S_2)\sin\theta d\theta \ , \tag{8}$$

where ψ_2 is the maximum value of θ for which rays pass through both spheres. As may be seen from Fig. 3, this happens when

$$d = D\sin\theta > R_1 + R_2 \ , \tag{9a}$$

which gives

$$\psi_2 = \arcsin\frac{R_1 + R_2}{D} \ . \tag{9b}$$

On the other hand, when

$$d = D\sin\theta < R_2 - R_1 \ , \tag{10a}$$

i.e., for

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$$\theta < \psi_1 = \arcsin\frac{R_2 - R_1}{D} \quad , \tag{10b}$$

the first circle lies completely inside the second, as shown in Fig. 4, and hence, $S_1 + S_2$ is a constant (namely, πR_1^2), and the integral from 0 to ψ_1 is easily evaluated to give

$$\frac{(1-\cos\psi_1)}{2} \quad . \tag{11}$$

Finally, the expression for $F_{1\to 2}$ is obtained by adding the contributions to the integral from 0 to ψ_1 and from ψ_1 to ψ_2 :

$$\begin{aligned} F_{1\to 2} &= \frac{(1-\cos\psi_1)}{2} + \frac{1}{2\pi R_1^2}\int_{\psi_1}^{\psi_2}(S_1+S_2)\sin\theta d\theta \\ &= \frac{(1-\cos\psi_1)}{2} + \frac{1}{2\pi R_1^2}(I_1+I_2) \quad . \end{aligned} \tag{12}$$

Using $(\pi/2 - \arcsin(\alpha/R_1)) = \arccos(\alpha/R_1)$ and $(\pi/2 - \arcsin(\gamma/R_2)) = \arccos(\gamma/R_2)$, we may write the integrals I_1 and I_2 as follows:

$$I_1 = R_1^2\int_{\psi_1}^{\psi_2}\left[\arccos\left(\frac{D^2\sin^2\theta + R_1^2 - R_2^2}{2DR_1\sin\theta}\right) - \frac{D^2\sin^2\theta + R_1^2 - R_2^2}{2DR_1\sin\theta}\sqrt{1-\left(\frac{D^2\sin^2\theta + R_1^2 - R_2^2}{2DR_1\sin\theta}\right)^2}\right]\sin\theta d\theta \tag{13a}$$

$$I_2 = R_2^2\int_{\psi_1}^{\psi_2}\left[\arccos\left(\frac{D^2\sin^2\theta + R_2^2 - R_1^2}{2DR_2\sin\theta}\right) - \frac{D^2\sin^2\theta + R_2^2 - R_1^2}{2DR_2\sin\theta}\sqrt{1-\left(\frac{D^2\sin^2\theta + R_2^2 - R_1^2}{2DR_2\sin\theta}\right)^2}\right]\sin\theta\, d\theta \quad . \tag{13b}$$

III.A. Spheres of Equal Radii

For the special case when $R_1 = R_2$, $\gamma = \alpha = (D\sin\theta)/2$ and $S_1 = S_2$. Moreover, $\psi_1 = 0$ and $\psi_2 = \arcsin(2R_1/D)$, and the expression for $F_{1\to 2}$ simplifies to

$$F_{1\to 2} = \frac{1}{\pi R_1^2}\int_0^{\sin^{-1}2R_1/D} S_1\sin\theta = \frac{1}{\pi R_1^2}\int_0^{\sin^{-1}2R_1/D} R_1^2\left[\left[\cos^{-1}\left(\frac{D\sin\theta}{2R_1}\right)\right] - D\sin\theta/2R_1\sqrt{1-(D\sin\theta/2R_1)^2}\right]\sin\theta d\theta = \frac{a^2}{\pi}\int_0^1\frac{u\cos^{-1}(u) - u^2\sqrt{1-u^2}}{\sqrt{1-a^2u^2}}du \tag{14}$$

where $a^2 = \left(\frac{2R_1}{D}\right)^2$, and we have made the substitution $u = (\sin\theta)/a$. For a =1, that is, for two spheres of equal radii touching each other, it becomes an elementary integral, and we get the result

$$F_{1\to 2} = \frac{1}{2} - \frac{4}{3\pi} \cong 0.0755868 \quad . \tag{15}$$

More generally, for nontouching spheres, the view factor $F_{1\to 2}$ can be written in terms of the complete elliptic integrals of the first and second kinds as follows:

$$F_{1\to 2} = \frac{1}{2} - \frac{2(a^2-1)K(a^2) + 2(a^2+1)E(a^2)}{3\pi a^2} \quad , \tag{16}$$

where

$$K(a^2) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-a^2\sin^2\theta}} = \int_0^1\frac{dt}{\sqrt{1-t^2}\sqrt{1-a^2t^2}}$$

and

$$E(a^2) = \int_0^{\pi/2}\sqrt{1-a^2\sin^2\theta}d\theta = \int_0^1\frac{\sqrt{1-a^2t^2}}{\sqrt{1-t^2}}dt$$

are complete elliptic integrals of the first and second kinds, respectively.

TABLE I

The View Factors Calculated Using Gauss Quadrature for the General Case and Eq. (16) for $R_1 = R_2$*

			$F_{1\to2}$		
R_1	R_2	D	Felske's Approximation	Exact [Eq. (16)]	Eight-Point Gauss Quadrature
1	1	2	0.0717968	0.0755868	0.0755869
1	1	3	0.0294373	0.0295903	0.0295899
1	1	5	0.0102051	0.0102108	0.0102106
1	1	10	0.0025126	0.0025127	0.0025126
1	2	3	0.1310700	–	0.1374081
1	2	5	0.0421684	–	0.0422752
1	2	10	0.0101274	–	0.0101288
1	4	5	0.2020410	–	0.2097192
1	4	10	0.0418473	–	0.0418733
1	4	20	0.0101084	–	0.0101087
1	9	10	0.2827637	–	0.2899321
1	9	20	0.0535192	–	0.0535303
1	9	50	0.0081675	–	0.0081676

*For the purpose of comparison, we also show the results obtained using the approximate formula given by Felske.

III.B. The General Case

In the general case, since the integrand is a smooth function of θ, we can easily calculate it using Gauss quadrature. The results using eight-point Gauss quadrature are shown in Table I. For the purpose of comparison, we also show the results obtained using the approximate formula given by Felske[1] and the formula obtained above for $R_1 = R_2$ [Eq. (16)] using Wolfram alpha. Table I shows that the eight-point Gauss quadrature results are practically the same as the exact ones (wherever available) to the order of accuracy shown in the table. On the whole, this simple method yields results that are much better than those obtained using Felske's approximate formula, particularly when the spheres are close to one another.

IV. CONCLUSION

We have reduced the four-dimensional integral giving the view factor for two spheres to a one-dimensional integral over the polar angle with respect to the line joining the centers of the spheres. For two equal spheres, the integral yields a compact formula in terms of complete elliptic integrals of the first and second kinds. For the general case, accurate integration is easily performed using low-order Gauss quadrature. Calculations have shown that an eight-point Gauss quadrature gives results that are close to the analytical results and are much better than Felske's approximation widely used for this purpose.

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