LETTER TO THE EDITOR

REPLY TO "RESPONSE TO 'COMMENTS ON "ELECTRON TRANSITIONS ON DEEP DIRAC LEVELS I"'"

Despite the rhetoric of Maly and Vávra[1] (MV), who used unwarranted terms like "confused" and "completely in error" in referring to our comments,[2] we are sincerely desirous of a rational and congenial resolution of this important scientific question. We would like to clarify the issues so that interested readers can easily decide for themselves and, more importantly, so that this important scientific issue can be decided upon by merit rather than by rhetoric.

The existence of deeply bound electron energy levels was proposed by MV in Ref. 3. According to MV, these solutions arise from previously neglected solutions of the relativistic Schrödinger and Dirac equations. In Ref. 2, we show that these deeply bound energy levels are physically impossible and arise from an incomplete analysis of the differential equations involved. In Ref. 1, MV claim that our analysis and conclusions are in error. This is our response to Ref. 1.

In Ref. 2, we analyze the relativistic Schrödinger equation for a potential that is a very simple model for a realistic nuclear system, with a constant interaction for $r \leq a$, where a is the nuclear radius, and an attractive Coulomb interaction for $r \geq a$. We obtain and approximate wavefunction (valid for small a), which is proportional to r in the interior region and which is a linear combination of solutions that have the same forms as the deeply bound and normal wavefunctions obtained for the Coulomb interaction alone. We then show that when the logarithmic derivatives of the interior and exterior solutions are matched at the boundary $r = a$, the coefficient corresponding to the deeply bound wavefunction is very small and, hence, does not significantly change the binding energy of the normal state.

The objections of MV in Ref. 1 are as follows:

1. The wavefunction $\chi(r) = r\psi(r) = B\rho^{s_+ + 1} + C\rho^{s_- + 1}$ cannot be used as the general exterior solution.

2. The correct interior wavefunction is $\chi(r) = A \sin Kr$, not AKr.

3. The variable $\rho = \alpha(E)r$ contains E_{s_+} or E_{s_-} inside $\alpha(E)$.

4. The coefficients B and C in the exterior wavefunction cannot be related through matching the interior and exterior wavefunctions.

We address these objections in order as follows.

1. Because the relativistic Schrödinger equation is a second-order differential equation, there are two linearly independent solutions. We may obtain the leading behavior (near the origin) of these solutions by analyzing the equation near $r = 0$. The relativistic Schrödinger equation near $r = 0$ is

$$\frac{d^2\chi}{d\rho^2} + \left[\frac{\gamma^2}{\rho^2}\right] = 0 \ , \tag{1}$$

and we may obtain the leading behavior of the independent solutions by substituting $\chi = \rho^{s+1}$ into Eq. (1). In this way, we obtain the indicial equation

$$s(s + 1) = -\gamma^2 \ , \tag{2}$$

which has the solutions

$$s_\pm = -\tfrac{1}{2} \pm (\tfrac{1}{4} - \gamma^2)^{1/2} \ . \tag{3}$$

Therefore, any solution to Eq. (1) has the leading behavior

$$\chi = B\rho^{s_+ + 1} + C\rho^{s_- + 1} \tag{4}$$

near the origin, where B and C are constants to be obtained from matching the interior and exterior solutions.

2. In Ref. 2, we specifically state that the form of the wavefunction χ that is given "has the following form, as $a \to 0$:

$$\chi(r) = \begin{cases} AKr \ , & r < a \\ B\rho^{s_+ + 1} + C\rho^{s_- + 1} \ , & r \gtrsim a \ . \end{cases} \text{"} \tag{5}$$

The actual interior solution is indeed $\chi(r) = A \sin Kr$, which has the leading behavior $\chi(r) = AKr$ near the origin.

3. The energy is obtained in the current case from the boundary condition that the wavefunction is finite as $r \to \infty$. Because the potential assumed here is different from a point Coulomb potential, which has energy levels E_+ and E_-, the energy E will in general be different from E_+ and E_-. In any case, this is a moot point because the parameter $\alpha(E)$

may be absorbed into the constants A, B, and C in the interior and exterior solutions without loss of generality.

4. We have shown that the exterior solution involves two constants, B and C, which must be related through matching the interior solution at the nuclear surface. For the sake of argument, let us assume that $C = 0$ and match the interior and exterior solutions, as in Eq. (7) of Ref. 1. We emphasize here that we are assuming that a is small, so that the solution is given by Eq. (5). Then,

$$\left.\frac{1}{\chi}\frac{d\chi}{d\rho}\right|_{r=a^-} = \frac{1}{a} = \left.\frac{1}{\chi}\frac{d\chi}{d\rho}\right|_{r=a^+} = \frac{s_+ + 1}{a} \quad . \tag{6}$$

This equation is clearly nonsensical because we have already shown that $s_+ \neq 0$; therefore, we need the solution corresponding to s_- to satisfy the matching conditions. In this way, we can relate the coefficients B and C.

A similar problem (for the nonrelativistic Schrödinger equation) is described in Ref. 4, and Landau and Lifshitz's conclusions are similar to ours.

We originally raised two points. Our first point was that deep Dirac levels are not warranted by a solution of either the relativistic Schrödinger or Dirac equations. Our second point was that if such orbits did indeed exist, this would not circumvent the absence of the nuclear products (ash) problem because such small neutral particles would be expected to have a very high fusion rate.

In summary, it appears that MV (Ref. 1) agreed with us on the second point as they did not attack us on point two. However, they vociferously disagreed with us on the first point. We think we have now incontrovertibly shown that such levels do not result from the Coulomb potential even in the relativistic case as claimed by MV (Ref. 3).

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