tities which can be evaluated by

$$\delta k_1 = -\alpha_0 \left(\frac{\phi_2}{\phi_1}\right) \tag{15}$$

$$\delta k_2 = -\alpha_0 \left(\frac{\phi_1}{\phi_2}\right) \tag{16}$$

The computer values for gain and phase lag of the reactor transfer function with flux tilting $\phi_1/\phi_2 = 1.2$ present, which represents the most severe flux tilting possible in the UTR-10 reactor, were not significantly different from those for a reactor without flux tilting. Additional break frequencies are introduced into the reactor transfer function by the flux tilting, but they occur at about the same frequency for the terms in the numerator and denominator. Consequently, no major change in the gain or phase shift results. The significant break frequencies still occur at λ and β/l.

The significant conclusions to be drawn from this study are:

1. Flux tilting is possible in the UTR-10 reactor. However, both fuel regions exhibit the same stable reactor period.

2. Flux tilting can affect the worth of a control rod but, due to the limited range of tilting that is possible in the UTR-10 reactor, variations of rod worth due to flux tilting should not be significant. The maximum degree of flux tilting occurs when the regulating rod is completely withdrawn.

3. A flux ratio of unity is never attained in the UTR-10 reactor for an excess reactivity of 0.48%, when the reactor is "just critical." The ratio of ϕ_1 to ϕ_2 varies from 1.05 to 1.18 with the higher flux in the region adjacent to the regulating rod.

4. The open loop response of the UTR-10 reactor to a sinusoidal variation of coupling, with or without flux tilting present, is not significantly different than the response of a single region reactor to a sinusoidal variation of reactivity.

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The Convergence of the Equipoise Method

The difference equations corresponding to the group-diffusion method may be written in the form

$$L\varphi = \lambda M\varphi \tag{1}$$

Here φ is a flux-vector whose dimension N is equal to the number of groups multiplied by the number of interior points in the reactor. L and M are $N \times N$ matrices. L has nonpositive off-diagonal elements, while the diagonal is positive. Due to the facts that the absorption cross sections are nonnegative, and that one may describe any multiplicative transfer of neutrons from one group to another by means of the fission matrix M, the diagonal of L dominates vertically (with strict dominance for at least one column). M is a nonnegative matrix. For a more detailed description of these matrices, see for instance ref. *1*.

One may construct a sequence of splittings

$$L = A_j + (L - A_j), \qquad (j = 1, 2, \cdots) \tag{2}$$

with nonsingular matrices A_j, and define an iterative procedure

$$A_j\varphi_j = (A_j - L + \lambda_{j-1}M)\varphi_{j-1} \qquad (j = 1, 2, \cdots) \tag{3}$$

$$\lambda_j = f(\varphi_j) \tag{4}$$

for determining the numerically smallest eigenvalue λ and the corresponding positive eigenvector φ. Here it must be required that the function f in (4) is chosen so that the substitution $\varphi_j = \varphi$ yields $\lambda_j = \lambda$.

Now it is well known that the scheme (3) converges independently of the f-function chosen, if $A_j = L$, corresponding to one of the usually employed methods, the power method. For two- or three-dimensional calculations the matrix L is not easily solvable, and (3) is solved iteratively by means of the so-called inner iterations. In most practical cases (at least for a two-group model) the eigenvalues of the matrix $L^{-1}M$ lie in the vicinity of the positive part of the real axis (although examples of reactors with negative or complex eigenvalues may be found) (*2*). Then one may use an iterative scheme, a little more elaborate than (3), the so-called Chebyshev-method (see for instance *3*).

The versions of the Equipoise method known to the author of this letter (*4, 5*) all utilize a basic scheme of the form (3), (4) with $A_j \neq L$. Such a scheme may diverge (see the examples given below). But usually A_j is chosen so that the matrix $A_j^{-1}(A_j - L)$ has a small spectral radius. In this case (3) could be regarded as an effective first inner iteration to solve (1) in the usual way as described above, and one has a fair hope of success.

However, for $A_j \neq L$ the convergence of the scheme (3), (4) may depend on the function f. The choice made in the Equipoise codes,

$$\lambda_j = \frac{\mathbf{e}^T L\varphi_j}{\mathbf{e}^T M\varphi_j}, \tag{5}$$

where $\mathbf{e}^T = (1; 1; \cdots; 1)$, seems to be a good one. It ensures that λ_j is positive, if $\varphi_j > \mathbf{0}$, without using numerical values of possibly negative numbers, as some other methods would require. This is a definite advantage, as will be clear from the following.

It is now assumed (for simplicity) that $A_j^{-1} \geqq 0$ and $A_j - L + \lambda_{j-1} M \geqq 0$ for all j. The initial vector φ_0 is chosen positive, which ensures that all $\varphi_j \geqq \mathbf{0}$. If the scheme (3)–(5) converges, it must converge to the unique positive eigenvector φ. If it should diverge, there are various possibilities.

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First, φ_j may tend to be proportional to a nonnegative vector ψ satisfying

$$cA\psi = (A - L + lM)\psi \tag{6}$$

$$l = \frac{\mathbf{e}^T L\psi}{\mathbf{e}^T M\psi} \tag{7}$$

where $c \neq 1$ is a positive number, and A is a limit for A_j, as $j \to \infty$.

It is seen that (6), (7) require $\mathbf{e}^T A\psi = 0$, and therefore this type of divergence could be excluded in a large number of cases. If, for instance, $A_j = A$ is independent of j, $\mathbf{e}^T A\varphi_j = \mathbf{e}^T A\varphi_{j-1} = \mathbf{e}^T A\varphi_0$, and therefore positive, if A is chosen corresponding, e.g., to the Jacobi or the Gauss-Seidel method. Furthermore, it follows from the assumptions made that $A\varphi_j$ is nonnegative and $\|\varphi_j\|$ therefore bounded independently of j. That is, even if φ_j is normalized, it cannot approach a vector ψ with $\mathbf{e}^T A\psi = 0$.

If the f-function is not the Equipoise-type, it is generally not difficult to construct examples of divergence corresponding to (6).

Some other types of divergence are possible, even with the Equipoise-function, namely the cyclic ones, where the sequence of iterates tends to be periodic. One might suggest the possibility of a periodic sequence with a not too small period n, so that these n vectors seem to converge to a vector $\psi \neq \varphi$. However, from (3) follows

$$\|\varphi_j - \varphi_{j-1}\| \geqq (|||A_j|||)^{-1} \|(-L + \lambda_{j-1}M)\varphi_{j-1}\|, \tag{8}$$

so that φ_j "close" to φ_{j-1} implies $(-L + \lambda_{j-1}M)\varphi_{j-1}$ "close" to $\mathbf{0}$ and, generally φ_j "close" to φ. It is probable, therefore, that such a bad behavior is unlikely with a reasonably good convergence criterion.

As an example of a cyclic divergence with a period of two one many take a two-group zero-dimensional reactor model with the matrices

$$L = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{4} \end{pmatrix} \tag{9}$$

Here the choice $l_{21} = -l_{11}$ does not give strict diagonal dominance but simplifies the calculations considerably. Moreover the fission matrix M is rather unusual. But otherwise these matrices satisfy the conditions imposed above.

The method (3)-(5) diverges in the manner indicated, if for the example (9) A is chosen corresponding to the Jacobi method ($A = I$) and the calculation is started with a suitable guess, for example $\varphi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In this case the periodic sequence will be:

$$\varphi_{2j} = \begin{pmatrix} 1.228715 \\ 0.771285 \end{pmatrix}; \qquad \lambda_{2j} = 0.351732;$$
$$\varphi_{2j+1} = \begin{pmatrix} 0.2712855 \\ 1.7287145 \end{pmatrix}; \qquad \lambda_{2j+1} = 0.710768. \tag{10}$$

Even if φ_0 is chosen very close to φ, the sequence of iterates will, in this case, "converge" towards a periodic sequence.

As an example of cyclic divergence with a period of two for the Gauss-Seidel method one may take a four-group zero-dimensional reactor model with the matrices

$$L = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{11} & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -\frac{10}{11} & 0 & -1 & 1 \end{Bmatrix}; \quad M = \begin{Bmatrix} 0 & 1.1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{Bmatrix} \tag{11}$$

Corresponding to the Gauss-Seidel method the elements of A_j in and below the diagonal are equal to those of L, while the others are equal to zero.

It is now seen that λ_j calculated by means of (5) is a function of the second and fourth coordinates of φ_j only,

$$\lambda_j = \frac{\varphi_{j,2}}{1.1(\varphi_{j,2} + \varphi_{j,4})}, \tag{12}$$

while these coordinates are connected to the corresponding ones of φ_{j-1} by means of the formula

$$\begin{Bmatrix} \varphi_{j,2} \\ \varphi_{j,4} \end{Bmatrix} = \begin{Bmatrix} 0.1\lambda_{j-1} & 1 + \lambda_{j-1} \\ \lambda_{j-1} & 0.1\lambda_{j-1} \end{Bmatrix} \begin{Bmatrix} \varphi_{j-1.2} \\ \varphi_{j-1.4} \end{Bmatrix} \tag{13}$$

which corresponds to the two-cyclic Jacobi method case

$$L = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}; \qquad M = \begin{pmatrix} 0.1 & 1 \\ 1 & 0.1 \end{pmatrix} \tag{14}$$

It is not difficult, either, to construct examples of N-cyclic divergence for the Jacobi method. It is conjectured, for instance, that a stable cycle of this type is generated, if one chooses an L-matrix whose elements are 1 in the diagonal, -1 in the subdiagonal, and 0 everywhere else, while the M-matrix has ones in the first subdiagonal, a one in the right upper corner, rather small numbers in the diagonal, and zeros everywhere else.

The examples of divergence given here are a little unusual or, rather, most peculiar and should be regarded only as illustrations of what types of divergence one should look for. It is felt, however, that some convergence criteria ought to be established before one could fully trust the Equipoise-methods. In particular, one should investigate in which cases a cycle of rather long period is stable, since such a cycle, as mentioned above, may give pseudo convergence.

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