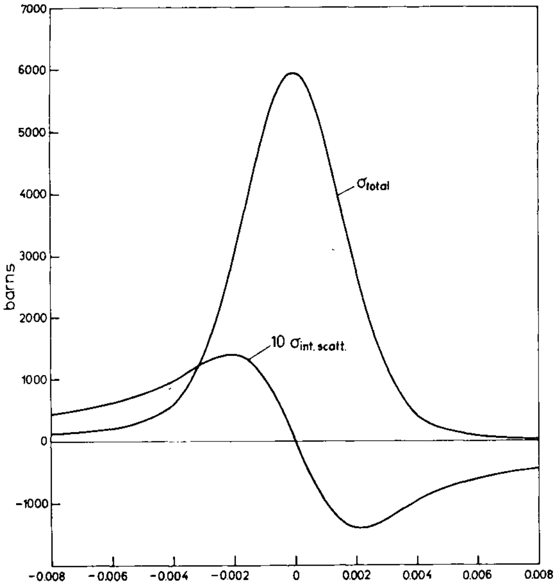

FIG. 1. Variation of total and interference scattering in the U^{238} resonance near the resonance energy $E_0 = 102.8$ ev for $T = 300°K$.

potential scattering cross section per uranium atom σ_p, as the potential scattering cross section of the uranium atom itself is equal to 10 barns. In all three equations considered, (2), (3), and (4) (exact, NR, and NRIA), the same numerical integration with the same lethargy steps is used. Taking the calculated values for two temperatures, the temperature coefficient of the effective resonance integral β

$$I(T) = I(T_0)[1 + \beta(\sqrt{T} - \sqrt{T_0})]. \tag{11}$$

The results in the columns denoted by NR-NRIA are obtained calculating the six broadest resonances (6.68, 21.0, 36.8, 66.3, 102.8, and 190.0 ev) according to the NRIA approximation and the remaining 49 resonances according to the NR approximation. Comparing the results of the NR and of the NR-NRIA approximations with the exact results for H_2O, it seems that the introduction of the NRIA approximation does not decrease very much the error of the pure NR approximation.

Comparing the exact resonance integrals for three different moderators at the same potential scattering cross section per uranium atom σ_p, one sees noticeable differences between them. As indicated in eq. (2), these differences are caused by different maximal lethargy losses per collision on moderator atoms $\epsilon_n = \ln 1/\alpha_n$ and by the different values of the scattering cross sections Σ_{sn} of the elements of light and heavy water. The effects are somewhat smaller if one also considers the interference between the potential and the resonance scattering. It is interesting to note that the exact temperature coefficient practically does not vary with the moderator properties but is noticeably affected by the interference scattering at small values of the potential scattering per uranium atom σ_p. As the increase of the resonance integral caused by the interference scattering is stronger at lower temperature, the temperature coefficient becomes smaller if one also takes into consideration the interference scattering. (Illustration for a resonance with relatively large interference scattering is given in Fig. 1.) The exact calculation compared with the NR approximation requires a reasonable increase of the machine time depending on the number of elements in the mixture.

We wish to express our gratitude to Mr. H. J. Siegert for carrying out calculations on a digital computer.

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Received January 15, 1962
Revised July 10, 1962

Velocity Dependent Neutron Transport Theory with High Energy Sources

In a recent paper Conkie (*1*) described a method of finding analytic approximate solutions to the Boltzmann equation dependent on both position and energy. Conkie's work was confined to the problem of thermal neutrons. In the present paper we wish to extend the method to the case where high energy neutrons are produced in the moderator. We use plane geometry and assume the moderator to be a slab occupying $|x| \leqq x_0$ and surrounded by vacuum. Heavy gas moderation is further assumed, with no capture, and with the neutron sources isotropic and monoenergetic and constantly distributed over the moderator.

The neutron velocities thus cover a broad interval. As in the paper by Conkie we are mainly interested here in obtaining the thermal neutron distribution. We therefore calculate the slowing down density at some velocity v_0 close to but greater than $\sqrt{2kT}$, and this density is then used to give the source term for the thermal region. The slowing down solution can be found in different ways. We use the well-known Greuling-Goertzel method (*2*). The source velocity is chosen to be $10^4\sqrt{2kT}$ and the moderator mass M is put equal to 3.6 to represent D_2O (*3*). The reason is that we wish to perform the calculations for a moderator mass different from 1 but not too big. Details of the calculations can be found in ref. 4. We find the distribution of neutrons at $v_0 = 3.355\sqrt{2kT}$ to be

$$N_S(x, v, \mu) = [F_0(x) + 3\mu F_1(x)]/2v_0^2 \tag{1}$$

where

$$\begin{aligned} F_0(x) = {} & 2.154 - 2\exp(-\lambda x_0) \\ & [0.6773x \sinh x - (2.140 + 0.6773x_0) \cosh \lambda x] \\ & - 22.36 \exp(-\lambda_1 x_0) \cosh \lambda_1 x + 2 \exp(-\lambda_2 x_0) \end{aligned} \tag{2}$$

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$$[2.915x \sinh \lambda_2 x + (7.101 - 2.915x_0) \cosh \lambda_2 x]$$

$$\begin{aligned} F_1(x) = {} & 2 \exp(-\lambda x_0) \\ & [0.5510x \cosh \lambda x - (1.344 + 0.5510x_0) \sinh \lambda x] \\ & + 8.708 \exp(-\lambda_1 x_0) \sinh \lambda_1 x - 2 \exp(-\lambda_2 x_0) \\ & [0.6570x \cosh \lambda_2 x + (2.889 - 0.6570x_0) \sinh \lambda_2 x] \end{aligned} \tag{3}$$

$$\lambda = 1.670, \qquad \lambda_1 = 0.8888, \qquad \lambda_2 = 0.5362 \tag{4}$$

Assuming now that (1) gives the distribution around v_0, the Boltzmann equation reads

$$\begin{aligned} v\mu \frac{\partial N(x, v, \mu)}{\partial x} & + F(v)N(x, v, \mu) \\ & = \int K(\mathbf{v}' \to \mathbf{v}) N(x, v', \mu')\, dv'\, d\mu' \\ & + \int_{v_0}^{(M+1/M-1)v} K_s(\mathbf{v}' \to \mathbf{v}) N_s(x, v', \mu')\, dv'\, d\mu' \end{aligned} \tag{5}$$

where K_s is the slowing down kernel and $K(\mathbf{v}' - \mathbf{v})$ is given by

$$K(\mathbf{v}' \to \mathbf{v}) = \frac{(M+1)^2}{4M^{3/2}} \frac{\sigma_0}{\pi^{3/2}} \frac{v^2}{|\mathbf{P}|} \exp\left[\frac{-M(v^2 - v'^2 + P^2/M)^2}{4P^2}\right]. \tag{6}$$

Here $\mathbf{P}$ is the momentum transfer and σ_0 the bound atom cross section.

In order to solve Eq. (5) we now adopt the following idea given by Conkie. The solution of the source-free equation is written

$$N(x, v, \mu) = \sum_j a_j \exp(g_j x) f_j(v, \mu) \tag{7}$$

where the μ-dependence is given by a spherical harmonics expansion and the v-dependence by a polynomial expansion.

If the symmetrized kernel is used it is easily shown that the $f_j(v, \mu)$ fulfil an orthogonality relation

$$\int dv\, d\mu v\mu f_j(v, \mu) f_k(v, \mu) = 0 \quad \text{for} \quad j \neq k. \tag{8}$$

This fact can be used to find a solution for the inhomogenous equation by the method of variation of constants. We thus try a solution of the form

$$N(x, v, \mu) = \sum_j a_j(x) \exp(g_j x) f_j(v, \mu) \tag{9}$$

giving for the determination of $a_j(x)$

$$\begin{aligned} \frac{da_j(x)}{dx} = {} & \exp(-g_j x) \int_{(M-1/M+1)v_0}^{v_0} dv \int_{-1}^{+1} d\mu f_j(v, \mu) \\ & \cdot \int_{v_0}^{(M+1/M-1)v} dv' \int_{-1}^{+1} d\mu' Ks(\mathbf{v}' \to \mathbf{v}) N_s(x, v', \mu'). \end{aligned} \tag{10}$$

The boundary conditions we use at the interface in our example are the same as Conkie's, that is, a modified version of Mark's boundary conditions. It should be noticed that the symmetry of the problem requires $N(x, v, \mu) = N(-x, v, -\mu)$.

We thus first have to determine the functions $f_j(v, \mu)$ in (7). This is done in a P_1-approximation and we write

$$f_j(v, \mu) = \tfrac{1}{2}[f_{j0}(v) + 3\mu f_{j1}(v)]. \tag{11}$$

For the velocity dependence we choose, like Conkie, a Tchebycheff polynomial expansion

$$\begin{aligned} f_{j0}(v) &= v \exp(-v^2/2) \sum_{m=0}^{9} b_{jm} T_m^*(y), \\ f_{j1}(v) &= v \exp(-v^2/2) \sum_{m=0}^{9} c_{jm} T_m^*(y) \end{aligned} \tag{12}$$

where $y = (v^2 + v^4)/(1 + v^2 + v^4)$. To obtain a P_1-approximation we have to calculate the moments of the kernel (6)

$$K(\mathbf{v}' - \mathbf{v}) = \tfrac{1}{2}[K_0(v' - v) + 3\mu K_1(v' - v)] \tag{13}$$

It seems that no explicit expression for $K_1(v' - v)$ has been published previously. The calculations may be found in (4).

As we have assumed no absorption, two of the eigenvalues g_j will be equal to zero and therefore the solution of the total equation (5), according to (9), is

$$\begin{aligned} N(x, v, \mu) = {} & \tfrac{1}{2}\{[A(x) + B(x)x] f_{00}(v) + 3\mu B(x) f_{01}(v)\} \\ & + \sum_{j=1}^{9} \{a_j(x) \exp(g_j x) f_j(v, \mu) \\ & + a_{-j}(x) \exp(-g_j x) f_{-j}(v, \mu)\} \end{aligned} \tag{14}$$

where

$$f_{-j}(v, \mu) = \tfrac{1}{2} f_{j0}(v) - \tfrac{3}{2}\mu f_{j1}(v). \tag{15}$$

The undetermined functions $A(x)$, $B(x)$, and $a_{\pm j}(x)$ are obtained according to (10). On account of the simple forms for the P_0- and P_1-moments of the slowing down kernel

$$\begin{aligned} K_{S0}(\mathbf{v}' \to \mathbf{v}) &= \frac{(M+1)^2}{2M} \cdot \frac{v}{v'} \\ K_{S1}(\mathbf{v}' \to \mathbf{v}) &= \frac{(M+1)^2}{2M} \cdot \frac{v}{v'} \left[\frac{M+1}{2} \frac{v}{v'} - \frac{M-1}{2} \frac{v'}{v}\right] \end{aligned} \tag{16}$$

and the slowing down distribution, it follows immediately that the integrations over v', μ, and μ' can be performed analytically without difficulty. The v-integration, however, is more complicated and one has to rely on a machine. It follows from (10) that the result depends on the choice of v_0. The integration over x, finally, is simple. The ten constants of integration have to be determined by the boundary conditions

$$N(x_0, v, \mu = -1/\sqrt{3}) = 0 \tag{17}$$

for all v, i.e., for all coefficients in the Tchebycheff polynomial expansion of $N(x, v, \mu)$. This gives ten equations. A further condition that will determine v_0 is that at this point of connection of the two energy regions the slowing down solution should fit continuously to the thermal solution.

The numerical calculations giving the solution of the homogenous equation were performed with the Chalk River Datatron using Simpson's rule to evaluate the integrals involved. As a result we obtain for the absolute values of the eigenvalues g_j:

0, 1.058, 4.504, 0.7927, 1.685, 1.834, 1.383, 2.110, 1.577, 1.754

The second lowest eigenvalue is probably too small, as one could expect that it should be greater than 1 in accordance with one-group spherical harmonics theory (5). It is interesting to compare with Conkie's result as our calculations have been done similarly. He also obtains one eigenvalue that is less than but quite close to one. The only difference between our calculations is that we here regard a moderator with a mass greater than one, which makes the kernel more complicated. However, as Conkie points out, it might be that the use of the P_1-approximation distorts the eigenvalue spectrum. The rapid oscillations in the higher Tchebycheff polynomials certainly makes it doubtful if the use of Simpson's rule is the best choice.

The equations (17) have been solved for $x_0 = 10$ and $x_0 = 100$ for different values of v_0 between 2.5 and 4 in order to make the difference between the thermal density and the slowing down density as small as possible. For $x_0 = 10$ it was found that the point of connection should be close to $v_0 = 2.96$ and for $x_0 = 100$, $v_0 = 3.50$ (the errors at these points are 0.003 and 0.009 respectively). We obtain at the interface

$$N_0(x_0, v) = v^2 \exp(-v^2) \sum_{k=0}^{9} a_k T_k^*(y) \qquad (18)$$

and we get for the coefficients a_k :

x_0	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
10	14.3	9.37	3.56	5.28	3.09	3.38	1.99	1.79	0.97	0.37
100	312.0	122.0	−30.2	31.6	−6.00	13.4	−0.70	6.12	0.18	1.68

For $x_0 = 100$ the convergence is good at the interface. Inside the slab, for $x = 0$, the density is a pure Maxwellian, given by

$$N_0(x = 0, v) = 30600v^2 \exp(-v^2), \qquad x_0 = 100. \qquad (19)$$

However, it is certainly much more interesting to obtain the neutron distribution for a slab that is only a couple of mean free paths thick. Unfortunately, even for $x_0 = 10$, the convergence is not good enough at the interface as is easily seen if one draws the corresponding curve, which shows small oscillations near the maximum. Conkie already remarked that the expansion variable used might not be the optimum choice and even the use of the Tchebycheff polynomials is somewhat arbitrary. It would also be convenient if one could use fewer terms in the expansion although that is not a major trouble with the big calculating machines. The problem of finding more suitable functions is now being considered and the hope is to let these be generated by the Boltzman$_n$ equation. There is another limitation to the method that will distort the distribution, and that is the P_1-approximation in μ. This should be of importance at the interface of thin slabs. In our case of $x_0 = 10$ the density inside the slab rapidly becomes convergent and it would therefore be worth while to try these calculations in a P_3-approximation. The computations are not very complicated, and once more suitable expansion functions have been found it should be possible to study more realistic problems, e.g., including absorption.

ACKNOWLEDGMENTS

This work started during a one-year visit to the Chalk River Project on an N.R.C. fellowship. It was suggested to me in a discussion with Drs. B. Davison, W. R. Conkie, and S. A. Kushneriuk, all of whom have given me considerable help and encouragement. I am further much indebted to Drs. L. G. Elliott and T. D. Newton for their kind interest and their efforts to make my visit at Chalk River a pleasant experience. My thanks are also due to the members of the Theoretical Physics Branch for many stimulating discussions and for help in performing the numerical calculations.

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A Note on the Measurement of the U^{238} Cadmium Ratio

It is customary to obtain the U^{238} cadmium ratio (CR) in a lattice by measuring the ratio of the U^{238} neutron capture induced activities in two uranium foils, irradiated bare and cadmium covered respectively in split fuel rods. The activity of the former is proportional to the sum of the thermal (subcadmium) and the epithermal (epicadmium) capture rates, the activity of the latter being proportional to the epithermal capture rate only.

It is well known that, because of the high U^{238} neutron resonance capture cross section, the epithermal neutron flux is more strongly depressed in the center of the fuel rods (and in the center of the foils) than is the thermal neutron flux. It follows that (roughly speaking) the thermal captures occur in the whole volume of the foils, whereas the epithermal captures are more concentrated near their outer perimeter. It was remarked some time ago (*1*) that the different spatial distributions of the capture in the two foils might be a source of error in the measurement of the CR. Self-absorption and geometry effects could cause the response of the counter to radiation produced near the foil periphery to be different from the response to radiation produced near the foil center. Nevertheless, recently published experimental results (*2*, *3*) obtained both by measuring foil activity directly and by homogenizing the foils first seem to agree reasonably well.

We have carried out a technique study with a light water moderated lattice of 1.15 wt. % U^{235} enriched metal rods of 0.387 in. diameter, at a 3:1 H_2O/U volume ratio. We have