Letters to the Editor

Comments on the Iterative Approach to a Space-Time Nonlinear Problem

In a recent paper on the solution of the nonlinear space-dependent neutron diffusion equation arising from an adiabatic excursion in a bare slab reactor,

$$\frac{\partial^2\varphi(x,t)}{\partial x^2} + B^2\varphi(x,t) - \alpha\left[\gamma \int_0^t \varphi(x,t')\,dt'\right]\varphi(x,t) = \left(\frac{H}{M}\right)^2 \frac{\partial\varphi(x,t)}{\partial t} \quad , \quad (1)$$

Shotkin[1] used an iterative scheme from which he calculated the energy released at the center of the slab [corresponding to $x = \frac{1}{2}$ since Eq. (1) is normalized to unity in the space dimension], by the relation

$$E_c = \gamma \int_0^\infty \varphi(\tfrac{1}{2}, t')\,dt' \quad . \quad (2)$$

From his solution of $\varphi(x,t)$ carried to the second iteration in which only the terms up to the spatial mode, $\sin 3\pi x$, was retained, he obtained his Eqs. (46) and (47) from which he calculated E_c. These two equations are shown below and denoted here as Eqs. (3) and (4).

$$\left[1 - \left(\frac{B}{\pi}\right)^2 + C_1 E\right] + \left[-\frac{C_3 D_1}{4} E^2 + \frac{C_3^2 G_1}{64} E^3\right] = 0 \quad , \quad (3)$$

$$E_c = \left[E + \frac{C_3}{8} E_2\right] + \left\{\frac{C_3}{64}\left[\left(\frac{B}{\pi}\right)^2 - 1\right] E^2 - \frac{C_3 D_3}{32} E^3 + \frac{C_3^2 G_3}{512} E^4\right\} \quad , \quad (4)$$

where

B^2 = the homogeneous material buckling after the step increase in reactivity

$C_1 = 8/(3\pi)$

$C_3 = -8/(15\pi)$

$D_1 = -8/(15\pi)$

$D_3 = 72/(35\pi)$

$G_1 = 72/(35\pi)$

$G_3 = 8/(9\pi)$.

We checked Shotkin's analysis of the adiabatic-excursion model up to Eqs. (3) and (4) and found these two equations to be correct. However his tabulated results for E_c calculated from Eqs. (3) and (4) were found to be erroneous. Table I shows Shotkin's and our results calculated from these two equations. Plotting the percentage difference (with the exact values[2]) vs $(B/\pi)^2$ we find that beyond about $(B/\pi)^2 = 5.0$ the percentage difference increases rapidly from ~ 1.0 to 8.9% at $(B/\pi)^2 = 10.0$. This differs greatly from Shotkin's calculated values, where the percent difference is only a maximum of 1.504% at $(B/\pi)^2 = 10.0$.

The second comment has to do with the order of magnitude of the last term in both Eqs. (3) and (4). The first and second square-bracketed terms in both Eqs. (3) and (4) were obtained by Shotkin in the first and second iterations, respectively. We noticed that even up to $(B/\pi)^2 = 10.0$ the magnitude of the last term in Eq. (4) is of much smaller order than the rest of the terms in the second bracket. This may be explained as follows.

Equation (1) was rewritten by Shotkin as

$$L_S\varphi = L_T\varphi + N(\varphi) \quad , \quad (5)$$

where

$$L_S = \frac{\partial^2}{\partial x^2} + \pi^2 \quad ,$$

$$L_T = \left(\frac{H}{M}\right)^2 \frac{\partial}{\partial t} - (B^2 - \pi^2) \quad ,$$

$$N(\varphi) = \alpha\gamma\varphi(x,t) \int_0^t \varphi(x,t')\,dt' \quad .$$

In our paper we now consider the whole right-hand side of Eq. (5) as a perturbation and we write

$$L_S\varphi = \epsilon[L_T\varphi + N(\varphi)] \quad , \quad (6)$$

$$\varphi = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \ldots \quad , \quad (7)$$

where we later set $\epsilon = 1$. The order of magnitude of the perturbation terms (θ_1, θ_2, etc.) is here denoted by ϵ. This is done to permit us to associate terms of the same order of magnitude.[3] As in Shotkin's paper we let

$$\theta_0(x,t) = A(t)\sin\pi x \quad .$$

We then have the following iterative scheme:

terms of order ϵ^0,

$$L_S\theta_0 = 0 \quad ,$$

terms of order ϵ^1,

$$L_S\,\theta_1 = L_T\theta_0 + \alpha\gamma\left[\theta_0 \int_0^t \theta_0\,dt'\right] \quad ,$$

terms of order ϵ^2,

$$L_S\theta_2 = L_T\theta_1 + \alpha\gamma\left[\theta_0 \int_0^t \theta_1\,dt' + \theta_1 \int_0^t \theta_0\,dt'\right] \quad ,$$

terms of order ϵ^3,

[1] L. M. SHOTKIN, *Nucl. Sci. Eng.*, **36**, 97 (1969).

[2] J. CANOSA, *Nucl. Sci. Eng.*, **32**, 156 (1968).

[3] R. BELLMAN, *Perturbation Techniques in Mathematics, Physics, and Engineering*, 1st ed., p. 2. Holt, Rinehart and Winston, Inc., New York (1964).

TABLE I

Iterative Solutions vs Exact Solution*

$$E_c = \gamma \int_0^\infty \varphi(\tfrac{1}{2},t')\,dt'$$

$$\text{Percent Difference} = \left|\frac{\text{Exact - Approximate}}{\text{Exact}}\right| \times 100$$

$(B/\pi)^2$	E_c	E_c [a]		E_c [b]		E_c [c]	
	Exact (Ref. 2)	Value	Percent Diff.	Value	Percent Diff.	Value	Percent Diff.
1.2	0.235	0.2349	0.085	0.2349		0.2349	
2	1.162	1.166	0.172	1.1616		1.1622	
3	2.297	2.306	0.391	2.2960		2.3002	0.051
4	3.408	3.395	0.381	3.4141	0.179	3.4276	0.575
7	6.640	6.637	0.045	6.8136	2.614	6.9311	4.384
8	7.693	7.720	0.376	8.0190	4.238	8.2224	6.881
10	9.774	9.921	1.504	10.6446	8.906	11.2024	14.615

*In Eq. (1) α is taken as $2\pi^2$ as in Ref. 2.
[a]Taken from Ref. 1 supposedly calculated from Eqs. (3) and (4).
[b]Values obtained by us using Eqs. (3) and (4). The values of E_c obtained here are correct to four decimal places.
[c]Values obtained from Eqs. (9) and (10).

$$L_S\theta_3 = L_T\theta_2 + \alpha\gamma\left[\theta_2\int_0^t \theta_0 dt' + \theta_1\int_0^t \theta_1 dt' + \theta_0\int_0^t \theta_2 dt'\right] \quad ,$$

etc.,

where the nonlinear terms are written explicitly. In Shotkin's iterative scheme the nonlinear term

$$\alpha\gamma\theta_1\int_0^t \theta_1 dt' \tag{8}$$

was included along with the terms

$$\alpha\gamma\theta_0\int_0^t \theta_1 dt' \text{ and } \alpha\gamma\theta_1\int_0^t \theta_0 dt'$$

in his second iteration. In our scheme the term (8) is of a different order as shown above. Since the last term of both Eqs. (3) and (4) arose from (8) this may explain why these last terms are of different order compared to the others in the second bracket of Eqs. (3) and (4).

In our scheme, if the term (8) is excluded we obtain, to order ϵ^2,

$$\left(\frac{B}{\pi}\right)^2 - 1 = C_1 E - \frac{C_3 D_1}{4} E^2 \quad , \tag{9}$$

$$E_c = E + \frac{C_3}{8}E^2 + \frac{C_3}{64}\left[\left(\frac{B}{\pi}\right)^2 - 1\right]E^2 - \frac{C_3 D_3}{32}E^3 \quad , \tag{10}$$

which are identical to Eqs. (3) and (4), respectively, except for the absence of the last terms. In this scheme the E-equation [that is Eq. (9)] is found by adding successively the coefficients of the $\sin\pi x$ mode from the right-hand side of all iterations up to the highest order iteration attempted and equating the sum to zero. This has its origin in the same argument used by Shotkin that the "secular" terms in the spatial mode expansion, that is terms in $\sin\pi x$ in the right-hand side of Eq. (5) [or Eq. (6) since $\epsilon = 1$], must be zero. Since Eq. (6) is broken-up into the different iteration equations, to obtain all secular terms up to the highest order iteration attempted, we must sum all these from the different iterations.

Table I shows the results calculated from Eqs. (9) and (10). We note that the percentage difference with the exact values is not drastically different from those calculated from Eqs. (3) and (4). Equations (9) and (10) still consider only the spatial modes up to $\sin 3\pi x$ in the iterations. In our scheme if a third order iteration is attempted it may be necessary to go up to $\sin 5\pi x$ mode and higher in all the iterations.

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Reply to Comments on the Iterative Approach to a Space-Time Nonlinear Problem

In response to the above Letter of Ibarra,[1] there was a numerical error on my part in Ref. 2. On examining my notes, I found that in calculating the quantity $C_3^2\, G_1/64$ in Eq. (46) of Ref. 2, I had correctly written $(0.16976)^2/64\ [72/(35\pi)]$ for the individual elements in this term but had somehow obtained 6.552×10^{-4} instead of the correct value 2.948×10^{-4}. On recalculation with this corrected value, I obtain agreement with the results of Ibarra.[1] These are shown in Table I, columns 2 and 3. These new values do not change any of the conclusions of Ref. 2. The one change that should be made (in addition to the corrections for the first iteration) is that in the paragraph after Eq. (47), the statement "The results . . . are seen to be in good agreement with the exact answers," should now read, "The results . . . are seen to be within 10% of the exact answers."

Although these percentage differences are an improvement on those obtained using a modal expansion,[3] they are

[1]H. IBARRA, "Comments on the Iterative Approach to a Space-Time Nonlinear Problem," *Nucl. Sci. Eng.*, **39**, 130 (1969).
[2]L. M. SHOTKIN, *Nucl. Sci. Eng.*, **36**, 97 (1969).
[3]J. CANOSA, *Nucl. Sci. Eng.*, **32**, 156 (1968).