Letters to the Editor

Comment on the Modeling of Thin Absorbers via δ-Functions

Since the early days of pile oscillator experiments, oscillators and control rods have been approximated in modeling and calculations by spatial δ-functions. The first use of this approximation is found in the pioneer work by Weinberg and Schweinler,[1] who used a three-dimensional δ-function to describe a small sample moved back and forth in a reactor. The use of one- and two-dimensional δ-functions in noise theory was introduced by Williams[2] to describe vibrating absorber plates. In all the works to which this letter refers, including this letter, one-group diffusion theory is used.

In all these cases, the space- and time- or frequency-dependent flux is calculated, or rather its deviation from the static flux (the fluctuating part of the response is searched). In such cases, there is no formal difficulty present when solving the relevant equations. This is because the equations are inhomogeneous; that is, they have the form

$$\hat{L}_t(r)\phi_t(r) = \delta(r - r_0) \ , \tag{1}$$

where $\hat{L}_t$ is a linear operator and both $\hat{L}_t$ and ϕ_t depend on time (or frequency). In dimensions higher than one, the solution diverges at the source point r_0, but this does not represent any major difficulty since the solution is discarded in points lying in close vicinity of the source point in view of the finiteness of the real oscillator or rod. The main point is, nevertheless, that it is possible to find a solution to Eq. (1) in the ordinary sense.

The situation changes when the static equations are considered. The equation will then be homogeneous; that is, one shall have

$$\hat{L}(r)\phi(r) = \gamma\phi(r)\delta(r - r_c) \ , \tag{2}$$

where ϕ stands for the static flux and $\hat{L}$ is the static (time-independent) diffusion operator. The constant γ is introduced to describe the strength of the absorber that is approximated by the δ-function, and it is equal to Galanin's thermal constant.[3]

In one dimension, that is, in a slab reactor with an absorbing plate, this equation still has an ordinary solution. Putting the rod (plate) at the center, one reads the equation as

$$\Delta\phi(x) + B^2\phi(x) = \frac{\gamma}{D}\,\delta(x)\phi(0) \ , \tag{3}$$

where

$$B^2 = \frac{\nu\sigma_f - \Sigma_a}{D}$$

and the parameters appearing on the right side of the foregoing equation have their usual meaning. In a reactor with extrapolated boundaries at $\pm a$, this has the solution

$$\phi(x) = A \sin B(a - |x|) \tag{4}$$

with the criticality equation[4]

$$\gamma = -2DB\cot(Ba) \ . \tag{5}$$

Solutions (4) and (5) can also be arrived at by starting with a rod of finite thickness d and absorption cross section Σ_a' and then letting $d \to 0$ and $\Sigma_a' \to \infty$ such that

$$d\Sigma_a' = \gamma \tag{6}$$

remains constant. This definition of γ leads to Eq. (3), and it also equals the definition of the thermal constant given by Galanin.[3]

In two dimensions, the situation is different. The two-dimensional diffusion equation, describing the static flux in a cylindrical reactor with axial symmetry and with an infinitely thin absorbing rod of thermal constant γ in the center, reads

$$\Delta\phi(r) + B^2\phi(r) = \frac{\gamma}{D}\,\delta(r)\phi(r) \ . \tag{7}$$

Making use of azimuthal symmetry, one can write Eq. (7) as

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi(r)}{dr}\right] + B^2\phi(r) = \frac{\gamma}{2\pi rD}\,\delta(r)\phi(0) \ . \tag{8}$$

For $r > 0$, Eq. (8) is a homogeneous equation, with a general solution that satisfies also the boundary condition $\phi(R) = 0$ as

$$\phi(r) = A[J_0(Br)Y_0(BR) - Y_0(Br)J_0(BR)] \ . \tag{9}$$

Equation (9) can be also written in the general form

$$\phi(r) = A[J_0(Br) + CY_0(Br)] \ . \tag{10}$$

Integration of Eq. (8) around the origin gives the condition

$$\lim_{\epsilon\to 0} r\frac{d\phi(r)}{dr}\bigg|_{r=\epsilon} = \frac{\gamma}{2\pi D}\,\phi(0) \ , \tag{11}$$

and substitution of solution (9) or (10) into condition (11) should yield a criticality condition between γ, B, and system size R in the form of an expression for C (which is also a function of BR).

There is no solution to Eq. (8) in the ordinary sense, that is, in the form of solution (10) with a finite C. A formal solution, satisfying the homogeneous form of Eq. (8) together with condition (11), can be given as follows:

$$\phi(r) = \lim_{a\to 0} A[J_0(Br) + C(a)Y_0(Br)] \ , \tag{12}$$

where the limit over $C(a)$ is defined such that

$$\phi(0) = \lim_{a\to 0}\left\{\lim_{r\to 0}[J_0(Br) + C(a)Y_0(Br)]\right\} = 0 \ . \tag{13}$$

This gives the condition

$$\lim_{a\to 0} C(a) = -\lim_{r\to 0}\frac{J_0(Br)}{Y_0(Br)} = 0 \ . \tag{14}$$

With these definitions, solution (12) will fulfill condition (11) because

$$\lim_{\epsilon\to 0} r\frac{d\phi(r)}{dr}\bigg|_{r=\epsilon} \sim -\lim_{\substack{\epsilon\to 0\\ a\to 0}} \epsilon C(a)\frac{1}{B\epsilon} = 0 = \frac{\gamma}{2\pi D}\phi(0) .$$

Solution (12) is not regular but is nonnegative and finite. For $r \neq 0$, it is equal to the unperturbed solution (solution without a rod), and for $r = 0$, it equals zero. From this, it also follows that the critical buckling must be equal to the B^2 of the unperturbed reactor in order to fulfill the boundary condition.

In other words, a δ-function absorber in two dimensions has no effect on the criticality of the system and on the static flux shape except at the rod position for the latter. The reaction rate and critical buckling is the same in the reactor with and without the rod.

The local and, with decreasing rod diameter, diminishing effect of a two-dimensional central cylindrical rod with radius a is well known.[5] It can also be seen by solving the corresponding equations for a finite rod. For simplicity, we describe the effect of the rod by prescribing a logarithmic boundary condition on its surface; i.e.,

$$\frac{\phi'(a)}{\phi(a)} = \gamma . \tag{15}$$

The γ above is equal to the inverse of the extrapolation distance into the rod, and an easy calculation in a gray rod yields that

$$\gamma = \frac{a\Sigma_a^r}{2} , \tag{16}$$

where again Σ_a^r is the absorption cross section of the rod. It can be also seen from either definition (15) or Eq. (16) that the γ here is $1/(2\pi a)$ times the thermal constant of Galanin[3] and Pázsit.[4] We use these two slightly different definitions of γ for the one- and two-dimensional cases, respectively, in the hope that they will not lead to confusion.

The solution outside the rod, satisfying the boundary condition at R, is given by Eq. (9), and substitution into condition (15) yields the criticality equation

$$\frac{J_0(BR)}{Y_0(BR)} = \frac{\gamma J_0(Ba) + BJ_1(Ba)}{\gamma Y_0(Ba) + BY_1(Ba)} . \tag{17}$$

This equation can serve to see how the transition to the singular solution (12) comes about with decreasing rod diameter a. The left side of Eq. (17), which gives the ratio between the coefficients of the $Y_0(Br)$ and the $J_0(Br)$ terms in solution (9), vanishes with vanishing rod radius a. The flux dip around the rod becomes more and more local but also steeper and steeper. At the same time, the critical B value approaches that of the unperturbed system.

The fact that the effect of an absorbing layer in a slab is global [see, e.g., Eq. (4)], whereas that of a cylindrical two-dimensional rod is local, was already remarked by Galanin.[3] His reasoning was that in a slab reactor, a neutron can cross the absorbing layer several times before it is finally absorbed or leaks out from the system; thus, the effect of the layer is enhanced and becomes global. In a different way, one can argue that in one-dimensional geometry, a neutron cannot pass from one side of the absorber to the other without passing through it, whereas in a two-dimensional case, it can get around the absorber rod.

Furthermore, in one dimension, letting d (absorber width) tend to zero while increasing Σ_a^r at the same rate will lead to a finite effect of the rod (layer) even in the limit of a δ-function because the total absorption rate of the rod (per unit y-z area) is preserved (assuming that the flux within the rod is unaffected by performing this limit, an assumption that can be shown to be true). On the other hand, keeping γ constant in Eq. (16) while letting $a \to 0$ (and thus increasing Σ_a^r) leads to a decreasing of the total absorption rate in the rod per unit axial length, even if the flux is unaffected by this procedure. This is because the area of the rod geometrical cross section diminishes quadratically with a (rod radius). Putting it differently, the probability of an average neutron path not crossing the rod will increase faster than the probability of capture within the rod of those neutrons whose path crosses the rod.

One might think that in the process of shrinking the rod into a δ-function, one could maintain a finite (bigger than zero) influence of the rod if the cross section were increased faster than $1/a$, that is, if one had $\gamma \to \infty$ sufficiently fast when $a \to 0$. However, Eq. (17) shows that this is not the case. No matter how fast γ diverges, the right side (and thus also the left side) of Eq. (17) vanishes, and thus, solution (9) reverts to that in the unperturbed system, with the effect of the rod on flux shape and criticality disappearing.

The explanation for this latter result is related to the fact that in a multiplying system of finite volume $V \in R^N$ of dimension $N = 1,2,3$, the effect of any finite absorbing subvolume $dV \in R^N$ remains finite even if the absorbing cross section of the subvolume dV diverges. This is due to self-shielding: With Σ_a diverging, the flux density diminishes almost everywhere in the subvolume dV, that is, except on its surface $dS \in R^{N-1}$. This way, since the flux remains nonzero only in a subdimensional volume, the total reaction rate

$$\lim_{\Sigma_a\to\infty} \int_{dV} \Sigma_a(r)\phi(r)\,dr \tag{18}$$

remains finite. The author cannot refer to any general proof of this statement, but the content of the statement feels intuitively plausible. An example of this in one dimension is given in Ref. 6, where the finite reactivity worth of an infinitely strong absorber is calculated. The foregoing results say furthermore that the integral in Eq. (18), that is, the effect of the absorber, vanishes if the volume dV tends to zero in two dimensions and higher dimensions.

We close this letter with one general and one specific comment on the modeling of small volumes via δ-functions that also offer an alternative explanation for the difference between the one-dimensional case and the case of higher dimensions. The general comment is that there is a self-contradiction in the process of decreasing a certain volume V and concurrently increasing a cross section, say Σ_a, ad infinitum. In a simple way, one can argue that with a given atomic density, increasing Σ_a means increasing the microscopic cross section σ_a. It is obviously a contradiction in terms to decrease V below $\sigma^{3/2}$, which is the volume corresponding to the (absorbing) cross section of one single (spherical) nucleus. The picture is obviously oversimplified but is used only in a rough qualitative way. The specific comment concerns the application of the above to explain the difference in one dimension and, say, two dimensions (finite effect and zero effect of a thin, strong absorber, respectively). In the one-dimensional case, the diffusion process takes place along the x axis, and to show up a sufficiently large cross section for the neutrons, the scattering or absorbing centers need to have a sufficiently large geometrical cross section in the y-z plane. Although completely unphysical, formally this is possible in a thin absorbing layer even if its thickness is tending to zero. One visualizes the host atoms as being smeared out into flat (two-dimensional) objects perpendicular to the x axis, in which process they can still maintain an arbitrarily large cross

section. In the two-dimensional case (cylindrical reactor with absorbing rod), however, the host atoms are smeared out into line filaments when the rod diameter tends to zero and, thus becoming one-dimensional, cannot have an arbitrarily large cross section (the cross sections actually vanish). Similarly, in three dimensions, a point absorber forces also the host atoms to become geometrical points. Thus, in two and three dimensions, the δ-function character of the absorbing volume overrides the assumption of the diverging cross sections, whereas in one dimension, it does not.

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Response to "Comment on 'Analysis of Cluster Geometries Using the DP1 Approximation of the $J_{\pm}$ Technique'"

There seems to be confusion between my definition of the $J_{\pm}$ technique and that of Mohanakrishnan.[1] In fact, what he calls the $J_{\pm}$ technique is what I would call the interface current method. The main difference between the two techniques is that the $J_{\pm}$ technique refers to a decomposition of a cell into isolated homogeneous zones, while the interface current method allows for a decomposition of the cell into heterogeneous zones. As a result, the computation of transmission probabilities is sufficient when the $J_{\pm}$ technique is considered, while the interface current method generally requires additional collision and leakage probabilities. However, for a given cell, the number of transmission probabilities required by the $J_{\pm}$ technique is generally much larger than that required by the interface current method. Since the purpose of my paper[2] was to discuss the use of the $J_{\pm}$ method, I did not think that a complete literature review of the interface current method was needed.

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