The average value of the neutron scattering cross section of beryllium is about 0.84 cm^{-1}, whereas the values of $\overline{\Sigma}_a$ do not exceed 0.93×10^{-2} cm^{-1}. (Except in one case in our second paper[9], where we have used $\overline{\Sigma}_a = 6.06 \times 10^{-2}$ cm^{-1}. This case was studied simply to investigate the effect of samarium resonances on the equilibrium spectra.) Thus, $\overline{\Sigma}_s$ is about a hundred times larger than $\overline{\Sigma}_a$ and we feel the use of diffusion theory by us was not unjustified.

Thus, though Michael has raised an important point, in view of what has been said above it is difficult to agree with his conclusions.

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[9]P. S. GROVER and L. S. KOTHARI, *Nucl. Sci. Eng.*, to be published.

Reply to a Note by Jeffery Lewins and a Simplified Development of the Maximum Principle

In our recent publication[1], the time-optimal solution to the xenon shutdown problem was obtained by application of the techniques of Pontryagin, Boltyanskii, Gamkrelidze, and Mishchenko[2]. An important consideration in formulating this problem was the choice of direction of the canonical (adjoint) vector in the xenon-iodine state space at the intersection of the optimal trajectory and the target curve, labeled Ω in Ref. 1. In his note, Lewins[3] incorrectly refers to this choice as a "supposition," whereas in Ref. 1, the unambiguous condition for the choice of sign of the xenon canonical variable p_2 in reverse time is specified by the statement, "The initial conditions at $T = 0$ [i.e., the intersection mentioned above] are determined by choosing a point x $(T = 0) \in \Omega$ according to Eq. (15) *and applying the additional conditions (11) and (18) . . .*" (Paragraph 1, page 474, Ref. 1). Equation (18) is the statement of transversality, and Eq. (11) prescribes the Hamiltonian, which is a positive constant[1,2]. Hence, we did not rely on the condition of transversality alone as suggested by Lewins. The desired manipulation of these *two* conditions is presented by Lewins in his equation (4). The same result follows easily from our equations (11) and (18), and it was for this reason that we stated in Ref. 1, "Equations (11) and (18) combine to specify $p(0)$ as an outwardly directed normal from Ω; i.e., $p_2(0) > 0$." (Paragraph 1, page 474 of Ref. 1).

We would also like to comment on a second statement in Lewin's note. In the paragraph containing his equation (4), he states, " . . . that since the bracket (in Eq. (4)) vanishes for operations on the xenon boundary, the sign is then immaterial and H is zero." This statement is puzzling since a) the bracket in his equation (4) refers to the intersection of the optimal trajectory with the target curve Ω, which has nothing to do with the xenon boundary, defined in Ref. 1 by $x_2 = x_{2\cdot max}$, and b) the corner conditions[2] require that H remain continuous at the intersection of an optimal trajectory with the boundary. Hence, H cannot equal zero on the boundary, since it is a positive constant off the boundary.

In the last paragraph of Lewin's note, he suggests changing the direction of the canonical variable and the optimization theorem to resolve an alledged conflict " . . . with our usual ideas of perturbation theory and the importance of a source of iodine or xenon." However, he adds that this will not affect our solution to the shutdown problem. Since we concur that the suggested change will leave the present results unaffected, we feel that there is no need for further comment.

Having dealt in detail with the specific comments of Lewins, we now return to the initial question regarding the sense of the adjoint vector p. We would like to present a simple geometric demonstration of the Maximum Principle for time optimal problems to show the manner in which the direction of p is related to the theory. The following development appeals to us as an excellent heuristic argument, but it is not to be construed as a rigorous derivation of either the Maximum Principle or the transversality condition. (We are indebted to Arthur M. Hopkin, University of California (Berkeley), for this model.)

In Fig. 1, let the initial point O be the origin. The target line is Ω. The contours S_T (assumed convex) enclose all points in the (x_1,x_2) phase plane that can be attained from O using any allowable control scheme during the time interval $O \leq t \leq T$. In Fig. 1, we observe:

a) the points on S_{t_1} can be reached in time t_1 only by employing time optimal control

b) if T is the minimum time from O to the target Ω, then S_T is tangent to Ω at the point where the target is attained.

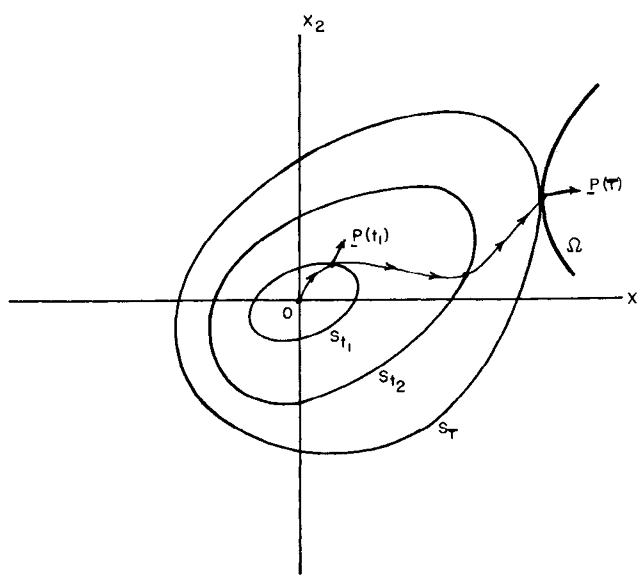

Fig. 1. Time-optimal trajectory from 0 to Ω. System equations:

$$\frac{dx}{dt} = f(x, u);\ 0 < t_1 < t_2 < T$$

for u in the allowable control space.

[1]J. J. ROBERTS and H. P. SMITH, Jr., "Time Optimal Solution to the Reactivity-Xenon Shutdown Problem," *Nucl. Sci. Eng.*, **22**, 470 (1965).
[2]L. S. PONTRYAGIN, V. G. BOLTYANSKII, R. G. GAMKRELIDZE and E. F. MISHCHENKO, *The Mathematical Theory of Optimal Processes*, (L. W. NEUSTADT, ed.), Interscience Publishers, New York, (1963).
[3]J. LEWINS, "A Note on the Adjoint Function in the Time Optimal Xenon Shutdown Problem," *Nucl. Sci. Eng.*, **23**, 404 (1965).

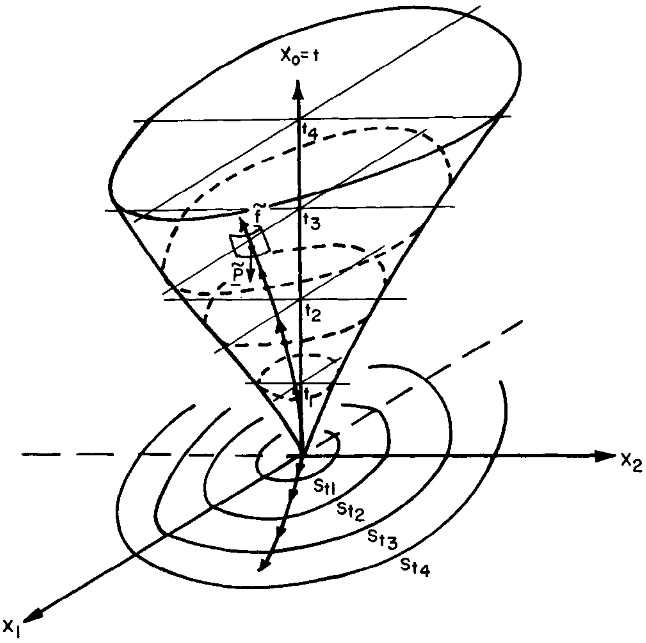

Fig. 2. "Cone of attainability" in the augmented phase space for a time-optimal second-order system.

$$\frac{dx}{dt} = \tilde{f} = (1, f_1, f_2)$$

$\tilde{p}$ ≡ outward normal to the surface.

The time-optimal trajectory lies on the surface. This is characterized by $(\tilde{f},\tilde{p}) = O$ along the trajectory.

We define an adjoint vector $p(t) = (p_1, p_2)$ as an *outward* normal to S_t (magnitude as yet unspecified)[4].

We define a reduced Hamiltonian (no cost function included) as

$$H_r = (p, f), \tag{1}$$

where $p(t)$ is the outward normal to S_t at the point where the optimal trajectory crosses S_t.

$$H_r \geqslant O \quad t \geqslant O \tag{2}$$

$$p \perp \Omega \quad t = T. \tag{3}$$

Equation (3) is the transversality condition. Note the direction of $p(T)$ relative to the target. Equation (2) is not the Maximum Principle, since H_r is not necessarily constant. For this aspect, we augment the space using the cost function, time, as the third coordinate (Fig. 2). By the arguments above, all points in the interior of the cone are attainable. Optimal trajectories that lay on S_t at time t now lie on the surface of the cone. If $\tilde{p} = (p_0, p_1, p_2)$ is the *outward* normal to the cone, then the Hamiltonian is now defined by

$$H = (\tilde{p}, \tilde{f}), \tag{4}$$

where

$$\tilde{f} = (1, f_1, f_2) = \tilde{f}(x, u).$$

For a trajectory to follow the surface, i.e,, to be optimal, we require $\tilde{f}$ tangent to the surface. This is the Maximum Principle:

$$H = O = \max_u (\tilde{p}, \tilde{f}). \tag{5}$$

The adjoint equation

$$dp/dt = -\partial H/\partial x \tag{6}$$

follows by assuming u fixed and setting $dH/dt = O$. Extension to other cost functions and higher dimensions is possible.

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[4] C. A. DESOER, "Pontryagin's Maximum Principle and the Principle of Optimality," *J. of Franklin Inst.*, **271**, 361 (1961).